Final Review

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$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0)$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$

Simplify the integrand if possible

Firstly, try to simplify the integrand if possible.

Example

$$\int \sqrt{x}(1+\sqrt{x})\,dx = \int (\sqrt{x}+x)\,dx = \cdots$$

Example

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta = \int \sin\theta\cos\theta\,d\theta = \cdots$$

$\mathcal{U}$-substitution

$\mathcal{U} = g(x)$ is in the integrand and its differential $d\mathcal{U} = g'(x)\,dx$ also occurs.

Example

$$\int x^2 e^{x^3}\,dx, \qquad \mathcal{U} = x^3, \quad d\mathcal{U} = 3x^2\,dx$$

Example

$$\int \frac{\ln x}{x}\,dx, \qquad \mathcal{U} = \ln x, \quad d\mathcal{U} = \frac{1}{x}\,dx$$

Integration by Parts: $\int U\,dV = UV - \int V\,dU$

Usually **two different types of functions** show up at the same time.

Example

$$\int x \sin x\,dx, \qquad U = x, \quad dV = \sin x\,dx$$

Example

$$\int x^2 e^x\,dx, \qquad U = x^2, \quad dV = e^x\,dx \quad (\textbf{Twice}\text{ I.B.P.})$$

Example

$$\int x \ln x\,dx, \qquad U = \ln x, \quad dV = x\,dx$$

Example

$$\int e^x \sin x\,dx, \qquad U = \sin x, \quad dV = e^x\,dx \quad (\textbf{Twice}\text{ I.B.P.})$$

Trigonometric Integrals

(i) Basic Trig. Definitions/Integral formulas & Pythagorean Identities

(ii) **Half** Angle and **Double** Angle Identities (or Formulas) $\cdots$ Use a lot!

(iii) $\sin(x)\cos(x)$ Integral Techniques & $\sec(x)\tan(x)$ Integral Techniques

Example

$$\int \sin^2(x)\cos^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,dx$$

Example

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx,\ \ u=\cos x,\ \ du=-\sin x\,dx$$

Example

$$\int \tan^2(x)\sec^2(x)\,dx,\quad u=\tan(x),\quad du=\sec^2(x)\,dx$$

Trigonometric Substitution

1. $\sqrt{a^2 - x^2}$, $x = a\sin\theta$ and use Identity $1 - \sin^2\theta = \cos^2\theta$.

Example

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx, \quad x = 3\sin\theta, \quad dx = 3\cos\theta\,d\theta$$

2. $\sqrt{a^2 + x^2}$, $x = a\tan\theta$ and use Identity $1 + \tan^2\theta = \sec^2\theta$.

Example

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx, \quad x = 2\tan\theta, \quad dx = 2\sec^2\theta\,d\theta$$

3. $\sqrt{x^2 - a^2}$, $x = a\sec\theta$ and use Identity $\sec^2\theta - 1 = \tan^2\theta$.

Example

$$\int \frac{1}{\sqrt{x^2 - 4}}\,dx, \quad x = 2\sec\theta, \quad dx = 2\sec\theta\tan\theta\,d\theta$$

Integration by Partial Fractions, I

Consider a rational function $\frac{P(x)}{Q(x)}$:

1. If $\deg(P(x)) \geq \deg(Q(x))$, do the long division calculation first:

Example

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

2. Factor the denominator $Q(x)$ as far as possible.
    - Linear factors (eg. $(x-r)^{m_L}$);
    - Irreducible quadratic factors (eg. $(x^2+px+q)^{m_Q}$, where $p^2 - 4q < 0$).

Example

$$Q(x) = x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

3. Two cases:

$$\sum_{i=1}^{m_L} \frac{A_i}{(x-r)^i}, \qquad \sum_{j=1}^{m_Q} \frac{B_j x + C_j}{(x^2+px+q)^j}.$$

Integration by Partial Fractions, II

Example

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Example

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Example

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Improper Integrals (of Type I/II)

Example

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x \Big|_1^t = \lim_{t\to\infty} (\ln t - 0) = \infty$$

Example

$$\int_0^{1} \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln x \Big|_t^1 = \lim_{t\to 0^+} (0 - \ln t) = \infty$$

Example

$$\int_0^{\infty} \frac{1}{x}\,dx = \int_0^1 \frac{1}{x}\,dx + \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx + \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

Remark.

Sometimes, L'Hôpital's Rule is helpful to evaluate the limits.

Chapter 10: Infinite Sequences and Series–Sequences

The main goal in this section is to study **Convergence of a sequence**.

(i) Limit Rules for Sequences: $(+, -, \times, \div$ and power rule)

(ii) The Sandwich Theorem for Sequences

(iii) The Continuous Function Theorem for Sequences (L'Hôpital's Rule)

(iv) The Monotonic Sequence Theorem

(v) Commonly Occurring Limits

Infinite Series, I

The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

is the **sequence of partial sums** of the series, the number S_n being the nth partial sum. The infinite series can be written as

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n.$$

Main question: Test convergence/divergence of the series.

Infinite Series, II

Theorem (Geometric series)

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases} \qquad (1)$$

Note that it's also calculable to find the sum of the **telescoping series.**

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

The inverse of theorem is not true in general. eg. *Harmonic Series* $\sum_{n=1}^{\infty} \frac{1}{n}$.

Theorem (The *n*th Term Test for Divergence)

If $\lim_{n\to\infty} a_n = L \neq 0$ *or fails to exist, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

The Integral Test

Theorem (The Integral Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a positive, continuous, decreasing function of x. Then the series

$$\sum_{n=N}^{\infty} a_n \text{ and the integral } \int_N^{\infty} f(x)\,dx \text{ both converge or diverge.}$$

Theorem (p-Series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Convergent} & \text{if } p > 1 \\ \text{Divergent} & \text{if } p \leq 1 \end{cases} \tag{2}$$

Remark.

$p = 1$: *Harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} \longleftrightarrow \int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b = \infty$ *diverges*

Comparison Tests

Theorem (Direct Comparison Test for Series)

If $0 \le a_n \le b_n$ *for all* $n \ge N$*, where* N *is a constant positive integer, then,*

1. *If* $\sum_{n=1}^{\infty} b_n$ *converges , then so does* $\sum_{n=1}^{\infty} a_n$*;*
2. *If* $\sum_{n=1}^{\infty} a_n$ *diverges , then so does* $\sum_{n=1}^{\infty} b_n$*.*

Theorem (Limit Comparison Test)

Suppose that $a_n > 0$ *and* $b_n > 0$ *for all* $n \ge N$ *(*N *an integer).*

1. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$*, then* $\sum a_n$ *and* $\sum b_n$ *both converge (or diverge).*
2. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
3. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Theorem (The Ratio Test: Important tool for power series)

Let $\sum a_n$ *be any series and suppose* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$,

1. *If* $L < 1$, *then the series* $\sum a_n$ *is absolutely convergent.*
2. *If* $L > 1$ *(including* $L = \infty$*), then the series* $\sum a_n$ *is divergent.*
3. *If* $L = 1$, *the Ratio Test is* **inconclusive**.

Theorem (The Root Test)

Let $\sum a_n$ *be any series and suppose* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$,

1. *If* $L < 1$, *then the series* $\sum a_n$ *is absolutely convergent.*
2. *If* $L > 1$ *(including* $L = \infty$*), then the series* $\sum a_n$ *is divergent.*
3. *If* $L = 1$, *the Root Test is* **inconclusive**.

Absolute Convergence vs. Conditional Convergence

Definition

A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Definition

We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges. *A classical example:* *Alternating Harmonic Series* $\sum (-1)^n \frac{1}{n}$.

Theorem (The Absolute Convergence Test)

If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

The Alternating Series Test

Theorem (The Alternating Series Test)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

1. *Nonincreasing:* $b_n \geq b_{n+1}$ *for all* $n \geq N$*, for some positive integer* N*,*
2. $\lim_{n\to\infty} b_n = 0$*.*

Example (The alternating p-series)

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p} = \begin{cases} \text{Absolutely Convergent} & \text{if } p > 1 \\ \text{Conditionally Convergent} & \text{if } 0 < p \leq 1 \\ \text{Divergent} & \text{if } p \leq 0 \end{cases} \tag{3}$$

Power Series

Theorem (The Radius of Convergence of a Power Series)

The convergence of the series $\sum c_n(x-a)^n$ *is one of the following 3 cases:*

1. *There is a positive number* R *such that the series diverges for* x *with* $|x-a|>R$ *but converges absolutely for* x *with* $|x-a|<R$. *The series may or may not converge at either of the endpoints* $x=a\pm R$.
2. *The series converges absolutely for every* x $(R=\infty)$
3. *The series converges only at* $x=a$ *and diverges elsewhere* $(R=0)$

R is called the *radius of convergence* of the power series, and the interval of radius R centered at $x=a$ is called the **interval of convergence**.

The interval of convergence may be open, closed or half open, depending on the series (*endpoints*).

How to test a Power Series for Convergence?

1. Use **Ratio** (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each **endpoint** ($|x - a| = R$). *Use a Comparison Test, the Integral Test, or the Alternating Series Test.*

3. If the interval of absolute convergence is $a - R < x < a + R$, the series **diverges** for $|x - a| > R$ (it does not even converge conditionally) *because the n^{th} term does not approach zero for those values of x.*

Operations on Power Series

(i) Addition/Subtraction of Power Series

(ii) Product of Power Series

(iii) Composition of a Power Series with a Continuous Function

Theorem (Substitution)

If $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for* $|x| < R$*, then*

$$\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$$

converges absolutely for any continuous function $f(x)$ *with* $|f(x)| < R$*.*

(iv) Term by Term **Differentiation** Theorem

(v) Term by Term **Integration** Theorem

Taylor and Maclaurin Series

Definition (Let $f(x)$ be ∞ly differentiable on an interval containing a)

The Taylor Series generated by $f(x)$ at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The Maclaurin Series of f is the Taylor series generated by f at $x = 0$, or

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

Note.

The Maclaurin series generated by f is often just called the Taylor series of f.

Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Applications of Taylor Series

1. Evaluating (or estimating) **Non-elementary Integrals**
2. Revisiting **Arctangents**
3. Evaluating **Indeterminate Forms**
4. Proving **Euler's Identity**

Common Taylor Series

1. $\frac{1}{1-x}$	$1+x+x^2+x^3+\cdots$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
2. $\frac{1}{1+x}$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\lvert x\rvert < 1$
3. e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\lvert x\rvert < \infty$
4. $\sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$\lvert x\rvert < \infty$
5. $\cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$	$\lvert x\rvert < \infty$
6. $\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$	$-1 < x \leq 1$
7. $\tan^{-1}(x)$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$	$\lvert x\rvert \leq 1$

Chapter 11: §11.1–Parametrisations of Plane Curves

(i) "Traveling Particle"

(ii) Cartesian Equations vs. Parametric Equations and Converting

(iii) Domains for the Parameter

(iv) Parametric equations for lines

(v) Parametric equations for circles

(vi) Parametric equations for parabola/hyperbola

(vii) Natural Parametrisations

§11.2–Calculus with Parametric Curves

(Parametric Formula for $\frac{dy}{dx}$) If all three derivatives exist and $\frac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \quad \cdots\cdots \dashrightarrow \begin{cases} \text{Tangent line equation} \\ \text{Area enclosed by curve} \end{cases}$$

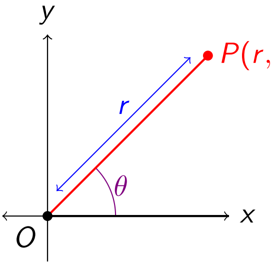
(Parametric Formula for $\frac{d^2y}{dx^2}$) Further we have $\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}$, if y is a twice-differentiable function of x.

Arc Length of Smooth Curves $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

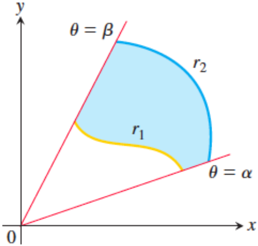
Revolution about the x-axis $(y \geq 0)$: $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Revolution about the y-axis $(x \geq 0)$: $S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

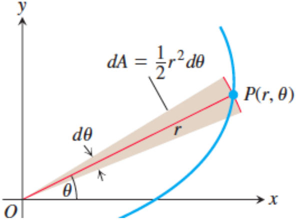
§11.3 & §11.5–Polar Coordinates



- Polar Equation and Graphs- - → vary r & θ resp.
- Relating Polar and Cartesian Coordinates
  $x = r\cos(\theta), \quad y = r\sin(\theta), \quad r^2 = x^2 + y^2.$



- Area in Polar Coordinates:



$$A = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta$$

$$A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\,d\theta$$

- Arc Lengths in Polar Coordinates: $L = \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$

Stay safe!

Good Luck for all Finals!!